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THE DYNAMIC STABILITY OF AN ELASTIC COLUMN*

S.D. FURTA

Lyapunov's second method is used to investigate the stability of the rectilinear equilibrium modes of a non-linearly elastic thin rod (column) compressed at its end. Stability here is implied relative to certain integral characteristics, of the type of norms in Sobolev spaces; the analysis is carried out for all values of the problem parameter except the bifurcation values.

The realm of problems connected with the Lagrange-Dirichlet equilibrium stability theorem and its converse involves specific difficulties when considered in the infinite-dimensional case: stability in infinite-dimensional systems is investigated relative to certain integral characteristics such as norms /1/, and as the latter may be chosen with a certain degree of arbitrariness, different choices may result in different stability results. On the other hand, there is no relaxation of any of the difficulties encountered in the case of a finite number of degrees of freedom.

We shall consider a certain natural mechanical system with a finite number of degrees of freedom. If the first non-trivial form of the potential energy expansion is positive-definite, the equilibrium position is stable. A similar statement has been proved for infinitely many dimensions as well /1-3/, using Lyapunov's direct method, and the total energy may play the role of the Lyapunov function.

The situation with respect to instability is more complex. In the finite-dimensional case, if the first non-trivial form of the potential energy expansion may take negative values, instability may be demonstrated in many cases by means of a function proposed by Chetayev in /4/. A general theorem has been proved /1/ for instability in infinitely many dimensions, relying on an analogue of Chetayev's function. Such functions have also been used /5, 6/ to prove the instability of equilibrium in specific linear systems with an infinite number of degrees of freedom.

However, Chetayev's functions /4/ are not suitable tools to prove the instability of equilibrium in most non-linear systems. Another "Chetayev function", which is actually a perturbed form of Chetayev's original function from /4/, has been proposed /7/, and it has been used to prove instability when the equilibrium position is an isolated critical point of the first non-trivial form of the potential energy expansion.

The majority of problems concerning the onset of instability of equilibrium configurations of elastic systems have been considered from a quasistatic point of view (see, e.g., /8, 9/). Problems of elastic stability and instability were considered in a dynamical setting in /2, 5/, where stability was investigated by Lyapunov's direct method. However, most of the results obtained in this branch of the field concern linear systems, and there are extremely few publications dealing with the onset of instability in non-linear elastic systems using Lyapunov's direct method. This is because in an unstable elastic system the quadratic part of the potential energy may change sign, and therefore the analogues of Chetayev's function from /4/ are not usually suitable for solving these problems. Dynamic instability has been studied or a specific non-linearly elastic system /10/, with the fact of instability established by using an analogue of the Chetayev function from /7/.

This paper presents one more example of a study of dynamic instability crried out for a non-linearly elastic system by Lyapunov's

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direct method.

1. We consider the motion of a non-linear elastic thin rod (column), with its ends attached by hinges, in the x, y plane. The shape of the rod is described by two functions $u(s, t), v(s, t)$ $(0 \leqslant s \leqslant 1, t \geqslant 0)$, representing non-dimensional displacements along the x and y axes.

In non-dimensional variables, the Lagrangian of the system becomes /11/

$$L = T - U; \quad T = {}^1/_2 \int_0^1 (u^{\cdot 2} + v^{\cdot 2})\, ds, \qquad U = {}^1/_2 \int_0^1 (v''^2 + \beta^{-1} (u' + {}^1/_2 v'^2)^2)\, ds \tag{1.1}$$

Here T and U are the kinetic and potential energies of the system, respectively, dots and primes denote partial derivatives with respect to t and s, respectively, and $\beta > 0$ is a constant. The functions $u(s, t)$ and $v(s, t)$ satisfy the boundary conditions

$$u(0, t) = -u(1, t) = c > 0, \ v(0, t) = v(1, t) = 0 \tag{1.2}$$

The constant $c > 0$ is known as the end contraction parameter.

To derive equations of motion for the system, we write down Hamilton's principle

$$\delta \int_0^T L\, dt = 0 \tag{1.3}$$

for variations $\delta u(s, t)$ and $\delta v(s, t)$ satisfying the conditions

$$\delta u(0, t) = \delta u(1, t) = \delta v(0, t) = \delta v(1, t) = 0, \qquad \delta u(s, 0) = \delta u(s, T) = \delta v(s, 0) = \delta v(s, T) = 0 \tag{1.4}$$

The equations of motion now follow from (1.3) and (1.4):

$$u^{\cdot\cdot} - \beta^{-1} (u' + {}^1/_2 v'^2)' = 0, \ v^{\cdot\cdot} + v^{\mathrm{IV}} - \beta^{-1} [v'(u' + {}^1/_2 v'^2)]' = 0 \tag{1.5}$$

with additional boundary conditions

$$v''(0, t) = v''(1, t) = 0 \tag{1.6}$$

It is obvious that Eqs.(1.5) with boundary conditions (1.2) and (1.6) have an equilibrium solution

$$u(s, t) = u_0(s) = c(1 - 2s), \ v(s, t) = 0 \tag{1.7}$$

Setting $u(s, t) = u_0(s) + w(s, t)$, we obtain a system of equations for the perturbed motion with boundary conditions

$$\begin{gathered} w^{\cdot\cdot} - \beta^{-1} w'' - {}^1/_2 \beta^{-1} (v'^2)' = 0 \\ v^{\cdot\cdot} + v^{\mathrm{IV}} + \lambda^2 v'' - \beta^{-1} [v'(w' + {}^1/_2 v'^2)]' = 0, \ \lambda = \sqrt{2c/\beta} \\ w(0, t) = w(1, t) = v(0, t) = v(1, t) = v''(0, t) = v''(1, t) = 0 \end{gathered} \tag{1.8}$$

We shall investigate the dynamic stability of the equilibrium modes (1.7) with respect to the parameter λ, i.e., the stability of the trivial solution of system (1.8)

$$w^{\cdot}(s, t) = v^{\cdot}(s, t) = w(s, t) = v(s, t) = 0$$

with respect to certain integral characteristics of the norm type.

2. Let us consider the Sobolev function spaces $\mathbf{W}_p^m [0, 1]$ /12/, $m \in \mathbf{N} \cup \{0\}$ (where $\mathbf{N}$ is the set of natural numbers), $p \geqslant 1$, $\mathbf{W}_p^0 [0, 1] = \mathbf{L}_p [0, 1]$, with norms

$$\| \varphi \|_{m, p} = (\int | \varphi^{(m)}(s) |^p ds)^{1/p} + (\int | \varphi(s) |^p ds)^{1/p} \tag{2.1}$$

Here $(\ \ )^{(m)}$ denotes the m-th generalized derivative; integration with respect to s is always over the interval $[0, 1]$.

It follows from (1.1) that a necessary condition for T and U to be finite is that

$$u^{\cdot}, v^{\cdot} \in \mathbf{W}_2^0 [0, 1], \ u \in \mathbf{W}_2^1 [0, 1], \ v \in \mathbf{W}_2^2 [0, 1] \cap \mathbf{W}_4^1 [0, 1], \quad \forall t \geqslant 0$$

By the Embedding Theorem $W_2^2[0,1] \subsetneq W_4^1[0,1]$ /12/, and therefore $v \in W_2^2[0,1]$.

Let $\varphi(s)$ be a function in $C^\infty[0,1]$ which satisfies certain boundary conditions $l_j[\varphi]=0$, $j=1,\ldots,J$, such that for some polynomial $P(s)$ of degree at most $m-1$, if $l_j[P]\equiv 0$ $(j=1,\ldots,J)$ then $P(s)\equiv 0$. Let $W_{2,0}^m[0,1]$ denote the closure of the linear space of such functions $\varphi \in C^\infty[0,1]$ in the norm (2.1) with $p=2$. As follows from /13/, the norm on $W_{2,0}^m[0,1]$ may be given by

$$\|\varphi\|_m = \left(\int [\varphi^{(m)}(s)]^2\, ds\right)^{1/2} \tag{2.2}$$

For any $t \geqslant 0$, the functions w, v are elements of the spaces $W_{2,0}^1[0,1]$, $W_{2,0}^2[0,1]$, for which the role of the boundary operators l_j $(j=1,\ldots,J)$ is played by the boundary conditions in system (1.8).

Let $\varphi \in W_{2,0}^m[0,1]$ and suppose that the system of functions $\sin \pi ks$, $k \in \mathbf{N}$, satisfies the appropriate boundary conditions $l_j = 0$ $(j=1,\ldots,J)$. Expand $\varphi(s)$ in Fourier series with respect to the function $\{\sin \pi ks\}_{k=1}^\infty$:

$$\varphi(s) = \Sigma \varphi_k \sin \pi ks, \quad \varphi_k = 2\int \varphi(s) \sin \pi ks\, ds$$

then it follows from (2.2) that

$$\|\varphi\|_m = \left(\Sigma (\pi k)^{2m}\varphi_k^2\right)^{1/2} / \sqrt{2} \tag{2.3}$$

Throughout, summation is performed from $k=1$ to $k=\infty$.

Let $\mathbf{F}^m$, $m \in \mathbf{Z}$ (where $\mathbf{Z}$ is the set of integers), denote the set of formal Fourier series

$$\varphi(s) = {}^1/_2\psi_0 + \Sigma(\varphi_k \sin \pi ks + \psi_k \cos \pi ks)$$

such that the norm

$$\|\varphi\|_m = \left(\psi_0^2 + \Sigma (\pi k)^{2m}(\varphi_k^2 + \psi_k^2)\right)^{1/2} / \sqrt{2} \tag{2.4}$$

is finite, and $\mathbf{F}_0^m \subset \mathbf{F}^m$ the subspace of elements such that $\psi_k = 0$, $k \in \mathbf{N} \cup \{0\}$.

Clearly, the spaces $\mathbf{F}_0^m$, $\mathbf{F}_0^{-m}$ are isomorphic to the spaces $W_{2,0}^m[0,1]$, $W_{2,0}^{-m}[0,1]$, where $W_{2,0}^{-m}[0,1]$ is the dual of $W_{2,0}^m[0,1]$. We observe, moreover, that $\mathbf{F}^n \subsetneq \mathbf{F}^m$ and $\|\cdot\|_m \leqslant \|\cdot\|_n$ whenever $n \geqslant m$.

Thus, we shall assume that $w^\cdot, v^\cdot \in W_{2,0}^0[0,1]$, $w \in W_{2,0}^1[0,1]$ $v \in W_{2,0}^2[0,1]$ for any $t \geqslant 0$, identifying these spaces with the corresponding subspaces of Fourier series $\mathbf{F}_2^m$. The boundary conditions in (1.8) are then automatically satisfied.

Replacing the norm-type integral characteristics (2.2) by equivalent expressions in terms of the Fourier coefficients (2.3) has certain advantages in the context of stability analysis; this was first pointed out in /14/.

Considering the phase space of the system, $W_{2,0}^0[0,1] \times W_{2,0}^0[0,1] \times W_{2,0}^1[0,1] \times W_{2,0}^2[0,1]$, we introduce the natural norm

$$\|(w^\cdot, v^\cdot, w, v)\|_* = \left(\|w^\cdot\|_0^2 + \|v^\cdot\|_0^2 + \|w\|_1^2 + \|v\|_2^2\right)^{1/2}$$

3. *Theorem 1.* If $\lambda \in (0,\pi)$, the trivial solution of (1.8) is stable relative to the norm $\|\cdot\|_*$.

Proof. The total energy functional is

$$H = T + U - U_0 = {}^1/_2\int \{w^{\cdot 2} + v^{\cdot 2} + v''^2 + {}^1/_2\beta^{-1}w'^2 - \lambda^2 v'^2 + {}^1/_4\beta^{-1}v'^4 + \beta^{-1}w'v'^2\}ds,\quad U_0 = {}^1/_2\beta^{-1}\int u_0'^2 ds \tag{3.1}$$

This functional is continuous in the norm $\|\cdot\|_*$. Indeed, for any $w_1^\cdot$, $v_1^\cdot$, w_1, v_1, $w_2^\cdot$, $v_2^\cdot$, w_2, v_2, the Cauchy inequality gives

$$\begin{gathered}|H(w_2^\cdot, v_2^\cdot, w_2, v_2) - H(w_1^\cdot, v_1^\cdot, w_1, v_1)| \leqslant \|w_2^\cdot + w_1^\cdot\|_0 \times \\ \|w_2^\cdot - w_1^\cdot\|_0 + \|v_2^\cdot + v_1^\cdot\|_0 \|v_2^\cdot - v_1^\cdot\|_0 + \|v_2'' + v_1''\|_0 \|v_2'' - \\ v_1''\|_0 + \lambda^2\|v_2' + v_1'\|_0\|v_2' - v_1'\|_0 + \beta^{-1}[\|w_2' + w_1'\|_0\|w_2' - \\ w_1'\|_0 + {}^1/_4(\|v_2'^3\|_0 + \|v_1'^3\|_0 + \|v_1'^4\|_0^{1/2}\|v_2'^2\|_0^{1/2} + \\ \|v_2'^4\|_0^{1/2}\|v_1'^2\|^{1/2}) \times \|v_2' - v_1'\|_0 + \|v_2'^2\|_0\|w_2' - w_1'\|_0 + \\ \|w_1'\|_0 \sup_{[0,1]}|v_1' + v_2'|\,\|v_2' - v_1'\|_0]\end{gathered}$$

If $w_1^\cdot$, $v_1^\cdot$, w_1, v_1, $w_2^\cdot$, $v_2^\cdot$, w_2, v_2, are bounded in the appropriate norms, the triangle inequality and the Embedding Theorem /12/ imply the following estimate:

$$|H(w_2^{\cdot}, v_2^{\cdot}, w_2, v_2) - H(w_1^{\cdot}, v_1^{\cdot}, w_1, v_1)| \leqslant C_1 (\| w_2^{\cdot} - w_1^{\cdot} \|_0 + \| v_2^{\cdot} - v_1^{\cdot} \|_0 + \| w_2 - w_1 \|_1 + \| v_2 - v_1 \|_2),\ C_1 > 0$$

which is equivalent to continuity in the norm $\|\cdot\|_*$.

Let us calculate the second variation of (3.1) at $w^{\cdot} = v^{\cdot} = w = v = 0$:

$$\delta^2 H = \int \{(\delta w^{\cdot})^2 + (\delta v^{\cdot})^2 + (\delta v'')^2 + \beta^{-1} (\delta w')^2 - \lambda^2 (\delta v')^2\}\, ds \tag{3.2}$$

Using (2.2)-(2.4), we get

$$\delta^2 H = \| \delta w^{\cdot} \|_0^2 + \| \delta w^{\cdot} \|_0^2 + \beta^{-1} \| \delta w \|_1^2 + {}^1/_2 \sum (k\pi)^2 ((k\pi)^2 - \lambda^2)(\delta v_k)^2 \geqslant \| \delta w^{\cdot} \|_0^2 + \| \delta v^{\cdot} \|_0^2 + \beta^{-1} \| \delta w \|_1^2 + C_2 \| \delta v \|_2^2$$

where δv_k are the coefficients of the Fourier expansion of $\delta v, 0 < C_2 < 1 - (\lambda/\pi)^2$. Consequently, the quadratic form (3.2) is positive-definite in the norm $\|\cdot\|_*$. Proceeding by analogy with the proof that H is continuous, one can prove similarly that H is twice continuously Fréchet-differentiable in the appropriate space, and therefore there exist constants $C_3, C_4 > 0$ such that

$$C_3 \| (w^{\cdot}, v^{\cdot}, w, v) \|_* \leqslant H \leqslant C_4 \| (w^{\cdot}, v^{\cdot}, w, v) \|_*$$

provided the perturbations $w^{\cdot}, v^{\cdot}, w, v$ are sufficiently small in the norm $\|\cdot\|_*$. Consequently, since the total energy of the system is a first integral, the functional (3.1) may be used in the capacity of a Lyapunov functional, so that, by Lyapunov's generalized stability theorem for equilibrium solutions of distributed systems /1, 3/, the solution of (1.8) is stable relative to the norm $\|\cdot\|_*$.

4. *Theorem 2.* If $\lambda \in (\pi, +\infty)$, $\lambda \neq \pi n$, $n \in \mathbf{N}$, then the solution $w^{\cdot} = v^{\cdot} = w = v = 0$ is stable relative to the norm $\|\cdot\|_*$.

Proof. Consider the infinite-dimensional analogue of Chetayev's function from /4, 7/:

$$W = \int (w^{\cdot}\Phi(w, v) + v^{\cdot}\Psi(w, v))\,ds \tag{4.1}$$

where $\Phi = w + \mu R_1 F$, $\Psi = v - \mu R_2 G$, $\mu > 0$ being a small parameter whose value will be determined later,

$$F = \nabla_w U_2 = -\beta^{-1} w'',\ G = \nabla_v U_2 = v^{\mathrm{IV}} + \lambda^2 v''$$

∇ denotes the Fréchet derivative, $U_2 = {}^1/_2 \int (v''^2 + \beta^{-1} w'^2 - \lambda^2 v'^2)\,ds$ is the quadratic part of the potential energy functional $U - U_0$. R_1, R_2 are selfadjoint Fredholm integral operators with kernels K_1, K_2,

$$R_1 F = \int K_1(s, \sigma) F(\sigma)\,d\sigma,\ R_2 G = \int K_2(s, \sigma) G(\sigma)\,d\sigma$$

$K_1(s, \sigma) = K_1(\sigma, s)$, $K_1(s, \sigma) = (s - 1)\sigma$, $\sigma \in [0, s]$, is Green's function of the second-order boundary-value problem

$$f'' = F,\ f(0) = f(1) = 0$$

$K_2(s, \sigma) = K_2(\sigma, s)$, $K_2(s, \sigma) = {}^1/_{12} [(s - \sigma)^3 + 2s\sigma (s^2 + \sigma^2 + 2) - (s + \sigma)^3]$, $\sigma \in [0, s]$, is Green's function of the fourth-order boundary-value problem

$$g^{\mathrm{IV}} = G,\ g(0) = g(1) = g''(0) = g''(1) = 0$$

Put $f = R_1 F$, $g = R_2 G$. These expressions are "smoothed" Fréchet gradients $\nabla_w U_2$, $\nabla_v U_2$, so (4.1) is an analogue of the Chetayev function proposed in /7/.

Note that the components of the Chetayev vector field (see /4, 7/) have the sense of virtual displacements, and therefore, the smoothing of F and G answers two purposes: first, Φ and Ψ must be of the same smoothness class as w and v; second, they must satisfy the same boundary conditions.

Suppose the contrary: the above equilibrium solution of system (1.8) is stable in Lyapunov's sense. Then for any fairly small (in the sense of the norm defined above) initial conditions $\| (w^{\cdot}, v^{\cdot}, w, v) \|_* < \varepsilon$ for $t \geqslant 0$ where $\varepsilon > 0$ is sufficiently small.

By the Cauchy inequality,

$$|W| \leqslant \| w^{\cdot} \|_0 \| \Phi \|_0 + \| v^{\cdot} \|_0 \| \Psi \|_0 \leqslant \varepsilon (\| \Phi \|_0 + \| \Psi \|_0)$$

We estimate the norms $\| \Phi \|_0$, $\| \Psi \|_0$

$$\| \Phi \|_0 \leqslant \| w \|_0 + \mu \| f \|_0 \leqslant \| w \|_1 + \mu \| f \|_0,\ \| \Psi \|_0 \leqslant \| v \|_0 + \mu \| g \|_0 \leqslant \| v \|_2 + \mu \| g \|_0$$

Obviously, $F \in W^{-1}_{2,0}[0,1]$, $G \in W^{-2}_{2,0}[0,1]$. Consider the Fourier expansions of the functions F, G, f, g in series of the functions $\{\sin \pi ks\}_{k=1}^{\infty}$:

$$F = \sum F_k \sin \pi ks,\ G = \sum G_k \sin \pi ks,\ f = \sum f_k \sin \pi ks,\ g = \sum g_k \sin \pi ks$$

then $f_k = -(\pi k)^{-2} F_k$, $g_k = (\pi k)^{-4} G_k$, which implies the equalities

$$\| f \|_0 = \| F \|_{-2},\ \| g \|_0 = \| G \|_{-4}$$

Applying (2.3), and (2.4), we obtain

$$\| F \|_{-2} \leqslant \beta^{-1} \| w \|_0 \leqslant \beta^{-1} \| w \|_1,\ \| G \|_{-4} \leqslant \| v \|_0 + \lambda^2 \| v \|_{-2} \leqslant (1+\lambda^2) \| v \|_2$$

Consequently, we have an estimate

$$| W | \leqslant M_1 \varepsilon^2 \tag{4.2}$$

where $M_1 > 0$ depends only on the parameters of the problem. Using a device analogous to that used in the previous section, one can show that the functional (4.1) is continuous in the norm $\| \cdot \|_*$.

We will now evaluate the derivative of (4.1) with respect to t along trajectories of Eqs.(1.8):

$$\begin{aligned} W^{\cdot} = & \int (w^{\cdot\cdot}\Phi + v^{\cdot\cdot}\Psi)\, ds + \int (w^{\cdot}\Phi^{\cdot} + v^{\cdot}\Psi^{\cdot}) ds = \\ & -\int (w\nabla_w U + v\nabla_v U)\, ds - \mu \int (f''f - g^{IV} g)\, ds - \\ & \mu \int [f\nabla_w (U - U_2) - g\nabla_v (U - U_2)]\, ds + \\ & \int (w^{\cdot 2} + v^{\cdot 2}) ds + \mu \int (w^{\cdot} f^{\cdot} - v^{\cdot} g^{\cdot}) ds \end{aligned} \tag{4.3}$$

Each of the integrals in this expression will now be estimated.

a) $-\int (w\nabla_w U + v\nabla_v U)\, ds = -2\,(U - U_0) - {}^1\!/_2 \beta^{-1} \int (w'v'^2 + {}^1\!/_2 v'^4) ds$

Using the Cauchy inequality, (2.1), (2.2) and the Embedding Theorem for $W_2^2[0,1]$ into $W_4^1[0,1]$ /12/, we get

$$\left| \int (v'^2 w' + {}^1\!/_2 v'^4) ds \right| \leqslant \| v \|^2_{1,4} (\| w \|_1 + {}^1\!/_2 \| v \|^2_{1,4})$$

and

$$-\int (w\nabla_w U + v\nabla_v U)\, ds \geqslant -2\,(U - U_0) - M_2 \| v \|_2^2\, (\| w \|_1 + \| v \|_2^2),\quad M_2 > 0 \tag{4.4}$$

b) $$-\mu \int (f''f - g^{IV} g)\, ds = \mu \int (f'^2 + g''^2) ds$$

where $f' = R_3 F$, $g'' = R_4 G$, R_1, R_2 are Fredholm integral operators with kernels K_3, K_4 :

$$R_3 F = \int K_3(s,\sigma) F(\sigma)\, d\sigma,\ R_4 G = \int K_4(s,\sigma) G(\sigma)\, d\sigma$$

$$K_3(s,\sigma) = \begin{cases} \sigma, & \sigma \in [0,s] \\ \sigma - 1, & \sigma \in (s,1] \end{cases},\quad K_4(s,\sigma) = K_1(s,\sigma)$$

Using the Fourier expansions of F, G, f, g, one can show that

$$\| f' \|_0^2 + \| g'' \|_0^2 = \| F \|_{-1}^2 + \| G \|_{-2}^2,\ \| F \|_{-1}^2 = \beta^{-2} \| w \|_1^2$$

Let $\lambda \in (\pi l, \pi(l+1))$, $l \in \mathbb{N}$. Expand the function v in Fourier series:

$$v = \sum v_k \sin \pi ks$$

$$\| G \|_{-2}^2 = {}^1\!/_2 \sum ((\pi k)^2 - \lambda^2)^2 v_k^2 \geqslant M_3(\lambda) \| v \|_2^2,$$

$$0 < M_3(\lambda) < \min \left\{ \left(1 - \left(\frac{\lambda}{\pi l}\right)^2\right)^2,\quad \left(1 - \left(\frac{\lambda}{\pi(l+1)}\right)^2\right)^2 \right\}$$

Consequently,

$$-\mu \int (f''f - g^{IV}g)\,ds \geqslant M_4(\lambda)(\|w\|_1^2 + \|v\|_2^2),\ M_4(\lambda) > 0 \tag{4.5}$$

c) Put $f_1' = \nabla_w (U - U_2) = -{}^1/_2\beta^{-1}(v'^2)'$, $\quad g_1' = \nabla_v (U - U_2) = -\beta^{-1}(v'(w' + {}^1/_2 v'^2))'$

Integrating by parts and applying the Cauchy inequality, we obtain

$$|\int (f_1'f - g_1'g)ds| = |\int (g_1g' - f_1f')ds| \leqslant \|g_1\|_0 \|g'\|_0 + \|f_1\|_0 \|f'\|_0,\ \|f'\|_0 = \beta^{-1}\|w\|_1,\ \|g'\|_0 = \|G\|_{-3} \leqslant \|G\|_{-2} \leqslant \|v\|_2 + \lambda^2\|v\|_0 \leqslant (1 + \lambda^2)\|v\|_2$$

By formulae (2.1)-(2.3) and the embedding theorems for $\mathbf{W}_2^2[0,1]$ into $\mathbf{W}_4^1[0,1]$, $\mathbf{W}_6^1[0,1]$ and $\mathbf{C}^1[0,1]$ /12/,

$$\|f_1\|_0 = {}^1/_2\beta^{-1}\|v'^2\|_0 \leqslant {}^1/_2\beta^{-1}\|v\|_{1,4}^2 \leqslant M_5\|v\|_2^2,\ M_5 > 0$$
$$\|g_1\|_0 \leqslant \beta^{-1}(\|v'w'\|_0 + {}^1/_2\|v'^3\|_0) \leqslant \beta^{-1}(\|w\|_1 \sup_{[0,1]} |v'| + {}^1/_2\|v\|_{1,6}^3) \leqslant M_6\|v\|_2(\|w\|_1 + \|v\|_2^2),\ M_6 > 0$$

Consequently,

$$-\mu \int [f\nabla_w (U - U_2) - g\nabla_v (U - U_2)]ds \geqslant -\mu M_7\|v\|_2^2(\|w\|_1 + \|v\|_2^2),\ M_7 > 0 \tag{4.6}$$

d) $\int (w^{\cdot 2} + v^{\cdot 2})ds = \|w^\cdot\|_0^2 + \|v^\cdot\|_0^2 \qquad (4.7)$

e) By the Cauchy inequality,

$$|\int (w^\cdot f^\cdot - v^\cdot g^\cdot)ds| \leqslant \|w^\cdot\|_0\|f^\cdot\|_0 + \|v^\cdot\|_0\|g^\cdot\|_0$$
$$f^\cdot = R_1F^\cdot,\ g^\cdot = R_2G^\cdot,\ F^\cdot = -\beta^{-1}w^{\cdot''},\ G^\cdot = v^{\cdot IV} + \lambda^2 v^{\cdot''}$$
$$\|f^\cdot\|_0 = \|F^\cdot\|_{-2} = \beta^{-1}\|w^\cdot\|_0,\ \|g^\cdot\|_0 = \|G^\cdot\|_{-4} \leqslant \|v^\cdot\|_0 + \lambda^2\|v^\cdot\|_{-2} \leqslant (1 + \lambda^2)\|v^\cdot\|_0$$

Consequently,

$$\mu \int (w^\cdot f^\cdot - v^\cdot g^\cdot)ds \geqslant -\mu M_8(\|w^\cdot\|_0^2 + \|v^\cdot\|_0^2),\ M_8 > 0 \tag{4.8}$$

Using (4.4)-(4.8), we estimate (4.3):

$$W^\cdot \geqslant -2(U - U_0) + \|v\|_2^2(\mu M_4(\lambda) - (M_2 + \mu M_7) \times (\|w\|_1 + \|v\|_2^2)) + (1 - \mu M_8)(\|w^\cdot\|_0^2 + \|v^\cdot\|_0^2) \tag{4.9}$$

Choose $\mu < 1/M_8$. Since $\lambda \neq \pi n,\ n \in \mathbf{N}$, it follows that $M_4(\lambda) > 0$. We shall assume that $\varepsilon > 0$ is so small that

$$\mu M_4(\lambda) - (M_2 + \mu M_7)(\varepsilon + \varepsilon^2) > 0$$

It then follows from (4.9) that

$$W^\cdot \geqslant -2(U - U_0)$$

If $\lambda > \pi$, then $U - U_0$ may become negative in an arbitrarily small neighbourhood of zero in $\mathbf{W}_{2,0}^1[0,1] \times \mathbf{W}_{2,0}^2[0,1]$. To prove this, it suffices to observe that the functional U_2 (the second variation at zero of the functional $U - U_0$) may take negative values. Indeed,

$$U_2(0, \sin \pi s) = \frac{\pi^2}{2}\int (\pi^2 \sin^2 \pi s - \lambda^2 \cos^2 \pi s)\,ds = \frac{\pi^2}{4}(\pi^2 - \lambda^2) < 0$$

Consequently, there may be motions with negative energy reserve $H = h < 0$ for any $t \geqslant 0$. Let $w(s,t), v(s,t)$ be a solution of (1.8) with initial conditions $w^\cdot(s,0) = 0$, $v^\cdot(s,0) = 0$, $w(s,0) = w_0(s)$, $v(s,0) = v_0(s)$, $\|(0, 0, w_0, v_0)\|_* < \varepsilon$ and energy reserve $h < 0$. Then for any $t \geqslant 0$ $W^\cdot \geqslant -2h$, whence it follows that $W \geqslant -2ht$, i.e., $W \to +\infty$ as $t \to +\infty$, contrary to the estimate (4.2), contradicting the assumed stability of the solution.

We have thus proved that the equilibrium (1.7) is unstable when $\lambda > \pi$, $\lambda \neq \pi n$, $n \in \mathbf{N}$.

It was shown in /11/ that when $\lambda > \pi$ the equations have equilibrium solutions other than the trivial one (1.7). It is known that if an elastic system has an adjacent equilibrium state, then the trivial mode is unstable, corroborating the result just stated. It should also be mentioned that $\lambda = \pi n,\ n \in \mathbf{N}$, are bifurcation values of the parameter /11/.

5. As remarked above, a different choice of integral characeristics may produce different stability results. From this point of view, a useful tool in proving that the trivial equilibrium (1.7) is unstable would be a smooth asmyptotic solution of (1.8), $w(s, t)$, $v(s, t)$, $w^{\cdot}(s, t)$, $v^{\cdot}(s, t)$, such that these functions tend to zero uniformly in s together with their partial derivatives with respect to s of any order as $t \to +\infty$. By virtue of the invariance of Eqs.(1.8) under time "reversal", this would mean that the trivial solution of (1.8) is unstable relative to any reasonably constructed integral characteristic. In the finite-dimensional case, asymptotic solutions of a non-linear sytem may be constructed by means of Lyapunov's first method /15/, which makes it possible to find such solutions in the form of exponential series, whose convergence is guaranteed by suitable theorems.

Let us determine conditions for the existence of a formal asymptotic solution of (1.8) in the form of exponential series:

$$w(s, t) = w_1(s)e^{-\alpha t} + w_2(s)e^{-2\alpha t} + \ldots,\ w_j(s) = \sum w_{jk} \sin \pi ks \tag{5.1}$$
$$v(s, t) = v_1(s)e^{-\alpha t} + v_2(s)e^{-2\alpha t} + \ldots,\ v_j(s) = \sum v_{jk} \sin \pi ks$$

The series (5.1)are analogous to these employed in Lyapunov's first method; if they were unfiormly convergent together with the corresponding series of partial derivatives, this would imply the existence of an asymptotic solution.

Theorem 3. If $\lambda > \pi$ and the quotient λ/π is irrational, Eqs.(1.8) have a formal asymptotic solution in the form (5.1).

Proof. Substituting (5.1) into (1.8), we obtain, on the left and right, series in integer powers of $e^{-\alpha t}$. Let us compare the coefficients of $e^{-j\alpha t}$.

For $j = 1$, we have

$$\alpha^2 w_1 - \beta^{-1} w_1'' = 0,\ \alpha^2 v_1 + v_1^{\mathrm{VI}} + \lambda^2 v_1'' = 0 \tag{5.2}$$

The equalities will hold if we set

$$w_1(s) = 0,\ v_1(s) = \sin \pi s,\ \alpha = \pi \sqrt{\lambda^2 - \pi^2}$$

For $j \geqslant 2$, we have

$$j^2\alpha^2 w_j - \beta^{-1} w_j'' = W_j,\ j^2\alpha^2 v_j + v_j^{\mathrm{IV}} + \lambda^2 v_j'' = V_j \tag{5.3}$$

where the functions $W_j(s), V_j(s)$ are polynomials in $w_1, \ldots, w_{j-1}, v_1, \ldots, v_{j-1}$ which, since they satisfy the boundary conditions, can be expanded in Fourier series

$$W_j(s) = \sum W_{jk} \sin \pi ks,\ V_j(s) = \sum V_{jk} \sin \pi ks$$

It then follows from (5.3) that

$$w_{jk} = W_{jk}/(j^2\alpha^2 + \beta^{-1}\pi^2 k^2) \tag{5.4}$$
$$v_{jk} = V_{jk}/(j^2\alpha^2 + k^2\pi^2(k^2\pi^2 - \lambda^2))$$

The condition $\lambda/\pi \notin \mathbf{Q}$ ($\mathbf{Q}$ (where Q is the set of rational numbers) guarantees that $j^2\alpha^2 + k^2\pi^2(k^2\pi^2 - \lambda^2) \neq 0$ for any $j, k \in \mathrm{N}$ and hence all the coefficients of the series (5.1) may be determined inductively.

The mechanical meaning of the condition $\lambda/\pi \notin \mathrm{Q}$ is that there is no exact integral resonance between the characteristic exponents of the system to a first approximation. In the finite-dimensional case one can avoid resonance phenomena while constructing asymptotic solutions by letting the generating solution be a solution of the linearized system with minimum characteristic exponent. However, the spectrum of an infinite-dimensional system is practically always unbounded, and therefore this device is hardly effective in the case in question. Indeed, it is evident from (5.4) that the problem of "small denominators" /16/ may be ecnountered in the attempt to determine the coefficients of the series (5.1), so that the question of whether these series are convergent is by no means trivial.

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FORCED OSCILLATIONS OF A NON-LINEAR SYSTEM WITH A REPULSIVE POSITIONAL FORCE*

A.A. ZEVIN and L.A. FILONENKO

Non-linear systems with one degree of freedom, in which the positional force is directed away from the equilibrium position of the system, are considered. The existence of forced periodic oscillations, their Lyapunov stability, and the behaviour of amplitude-frequency characteristics are investigated. It is shown that stable periodic oscillations are possible in the case when the positional force has non-monotonic properties. Forced oscillations of a pendulum with respect to the upper equilibrium position are considered as an example.

Systems with repulsive positional forces appear not to have been previously considered in the literature. The well-known analytical methods of non-linear mechanics (/1, 2/ etc.) are based on the assumption of the nearness of the solutions under investigation to solution of the corresponding autonomous system, and are inapplicable to our systems because there are no periodic generating solutions. In this paper a qualitative investigation is made of

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